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Abstract—A spectral–difference method is developed for the problem of planar filtrational convection of incompressible fluid in a porous medium (the Darcy model). Based on a spectral decomposition with respect to the vertical coordinate and differencing with respect to the horizontal coordinate, a finite-dimensional system of ordinary differential equations preserving the cosymmetry of the original problem is derived. A method for computing families of steady regimes is developed, and numerical results demonstrating the importance of cosymmetry preservation are presented.

INTRODUCTION

Recently, systems that exhibit continuous families of regimes were found, with spectra varying over the families. In particular, one-parameter families of steady convective regimes were obtained in the filtration problem for a viscous fluid in a porous medium governed by Darcy’s law [1, 2]. This phenomenon was explained by invoking the cosymmetry theory [2–6]. In a sense, the concept of cosymmetry is dual to that of symmetry. The existence of a nontrivial cosymmetry indicates that the system is underdetermined and there exists a hidden parameter, implying that families of steady-state solutions can be obtained. The principal distinction of cosymmetric dynamical systems from symmetric systems lies in the variability of their stability spectrum [2].

In developing numerical methods of mathematical physics, special attention should be focused on the preservation of properties of the original equations. The importance of retaining the conservative properties and conservation laws in finite-difference approximations was explained in [7, 8]. It was found that the cosymmetry of the original problem must be preserved in a finite-dimensional approximation in order to compute a family of steady-state solutions to cosymmetric partial differential equations. In [9], a finite-difference method involving calculation of convective terms based on the approximations developed in [10, 11] was applied to solve the problem of planar filtrational convection governed by Darcy’s law and to compute families of steady regimes. For discretizations that do not preserve cosymmetry, a set of isolated equilibria was obtained in computations instead of a continuous family of steady convective regimes.

Efficient tools for solving problems in mathematical physics are provided by spectral [12] and spectral–difference methods. In the latter, the spectral approach is applied to some spatial coordinates, whereas the remaining coordinates are treated by means of a finite-difference method. In particular, a spectral–difference method was used in [13, 14] to simulate turbulent incompressible pipe and channel flows and in [15] to compute seismic waves in elastic media.

In this paper, we consider the implementation of a spectral–difference method designed to solve the problem of planar filtrational convection governed by Darcy’s law and derive cosymmetry-preserving approximations. To compute families of steady regimes, one must repeatedly solve nonlinear systems of equations obtained as approximations of the starting partial differential equations. It was found that reduction of computational costs is facilitated by using an approach based on a spectral decomposition over a small number of harmonics in the vertical coordinate combined with a more detailed finite-difference discretization in the horizontal direction. A spectral decomposition with respect to both coordinates (Galerkin’s method) was applied to analyze the problem in [16–18]. Approximations with a small number of modes were used in [16], whereas the analysis of finite-dimensional systems with the number of harmonics varying from 36 to 81 conducted in [17, 18] made it possible to compute a family of steady-state solutions and predict the onset of instability for the family.
1. EQUATIONS OF GRAVITATIONAL CONVECTION IN A POROUS MEDIUM

The equations of gravitational convection [1, 2] are written in dimensionless variables as

\[
\frac{d\Theta}{dt} = \Delta \Theta + \lambda \Psi_x + J(\Psi, \Theta) \equiv F_1(\Theta, \Psi), \quad J(\Psi, \Theta) = \Psi_x \Theta_3 - \Psi_3 \Theta_x.
\]  

(1.1)

\[
0 = \Delta \Psi - \Theta_x = F_2(\Theta, \Psi).
\]  

(1.2)

Here, \(x\) and \(y\) are Cartesian coordinates on a plane, \(\Delta\) is the Laplacian, \(t\) is time, \(\Psi(x, y, t)\) is the streamfunction, \(\Theta(x, y, t)\) is the deviation of temperature from the equilibrium linear profile, and \(J(\Psi, \Theta)\) is the Jacobian. The parameter \(\lambda\) is the filtration Rayleigh number defined as \(\lambda = \beta g A k l / (\chi v)\), where \(\beta\) is the coefficient of thermal expansion, \(g\) is the gravitational acceleration, \(A\) is the characteristic temperature difference, \(k\) is permeability, \(l\) is a reference length, \(\chi\) is thermal diffusivity, and \(v\) is kinematic viscosity.

On the boundary of the rectangle \(D = [0, a] \times [0, b]\), Dirichlet boundary conditions are set:

\[
\Psi = 0, \quad \Theta = 0.
\]  

(1.3)

The initial condition for system (1.1), (1.2) has the form

\[
\Theta(x, y, 0) = \Theta_0(x, y),
\]  

(1.4)

where \(\Theta_0(x, y)\) is a function defined on \(D\).

The problem considered here is globally solvable and dissipative, and the quiescent state is globally stable when \(\lambda\) is small [2, 3]. For a constant \(\lambda\), the streamfunction \(\Psi\) can be expressed in terms of \(\Theta\) by solving the Dirichlet problem for Poisson’s equation (1.2). The result can be written as \(\Psi = G \Theta_0\), where \(G\) is an appropriate Green’s operator. It was shown in [2, 3] that the problem of planar filtrational convection governed by Darcy’s law possesses a cosymmetry. For (1.1)–(1.4), the cosymmetry is the vector function \(L = (\Psi, -\Theta)\). Therefore, multiplying (1.1) by \(\Psi\) and (1.2) by \(\Theta\) and adding the results, we obtain

\[
\int_D [F_1(\Theta, \Psi) \Psi - F_2(\Theta, \Psi) \Theta] dx dy = 0.
\]  

(1.5)

This is verified by performing direct integration by parts, applying Green’s theorem, and making use of the following property of the Jacobian \(J(\Psi, \Theta)\):

\[
\int_D J(\Psi, \Theta) \Psi dx dy = 0.
\]  

(1.6)

Moreover, the Jacobian \(J\) is antisymmetric with respect to its arguments, and

\[
\int_D J(\Psi, \Theta) \Theta dx dy = 0.
\]  

(1.7)

The stability of the quiescent state in this problem was analyzed in [1–3]. The critical values of \(\lambda\) are \(\lambda_{m_n} = (2\pi m/a)^2 + (2\pi n/b)^2\) \((m, n = 1, 2, \ldots)\). The first critical point \(\lambda_{1,1}\) is always doubly degenerate, and a cycle of stable steady regimes branches off from the quiescent state at \(\lambda = \lambda_{1,1}\). Each passage \(\lambda\) through a critical point \(\lambda_{m_n}\) is associated with a bifurcation giving rise to a cycle of unstable steady regimes [3].

2. SPECTRAL–DIFFERENCE METHOD

We seek a solution to problem (1.1)–(1.4) in the form of a sine series:

\[
\Theta(x, y, t) = \sum_{j=1}^{m} \Theta_j(x, t) \sin \frac{\pi j}{b} y, \quad \Psi(x, y, t) = \sum_{j=1}^{m} \Psi_j(x, t) \sin \frac{\pi j}{b} y.
\]  

(2.1)

Substituting (2.1) into Eqs. (1.1) and (1.2) and performing projections, we obtain the set of equations

\[
\dot{\Theta}_j = \Theta''_j - c_j \Theta_j + \lambda \Psi'_j - J_j \equiv \Phi_{1,j}, \quad j = 1, m,
\]  

(2.2)

\[
0 = \Psi''_j - c_j \Psi'_j - \Theta'_j \equiv \Phi_{2,j}, \quad j = 1, m.
\]  

(2.3)
Henceforth, the prime denotes a derivative with respect to $x$; the dot, a derivative with respect to $t$, $c_j = j^2 \pi^2/b^2$; and $J_j$ is expressed as
\[
J_j = \frac{2\pi}{b} \sum_{i=1}^{m-j} [(i+j)(\Theta_{i+j} \Psi_j - \Theta_j \Psi_{i+j}) + i(\Theta'_{i+j} \Psi_i - \Theta'_j \Psi_{i+j})]
\]
\[+ \frac{2\pi}{b} \sum_{i=1}^{j-1} (j-i)(\Theta'_{j-i} \Psi_j - \Theta'_i \Psi_j), \quad j = 1, m. \tag{2.4}
\]

Boundary conditions (1.3) are rewritten as
\[
\Theta_j(t, 0) = \Theta_j(t, a) = 0, \quad \Psi_j(t, 0) = \Psi_j(t, a) = 0, \quad j = 1, m.
\]

Initial condition (1.4) becomes
\[
\Theta_j(x, 0) = \int_0^y \frac{\pi y}{b} dy, \quad j = 1, m.
\]

To approximate Eqs. (2.2) and (2.3) with respect to $x$, we apply a finite-difference method. We define the grid $\Omega = \{x_k | x_k = kh, k = 0, n + 1, h = a/(n+1)\}$ on the segment $[0, a]$ and introduce
\[
f_{j,k} = \Theta_j(x_k, t), \quad g_{j,k} = \Psi_j(x_k, t), \quad J_{j,k} = J_j(x_k, t).
\]

The first and second derivatives in the linear parts of Eqs. (2.2) and (2.3) are approximated by central differences. As a result, we obtain the following system of ordinary differential equations:
\[
f_{j,k} = \frac{2f_{j,k+1} + f_{j,k-1} - c_j f_{j,k}}{h^2} - \frac{\lambda(g_{j,k+1} - g_{j,k-1})}{2h} - J_{j,k} = \Phi_{1,j,k},
\]
\[0 = \frac{2g_{j,k+1} + g_{j,k-1} - c_j g_{j,k}}{h^2} - \frac{f_{j,k+1} - f_{j,k-1}}{2h} = \Phi_{2,j,k}. \tag{2.5}
\]

The specific expression for the approximation $J_{j,k}$ of the function $J_j$ in (2.4) at a point $x_k$ is given below.

3. COSYMMETRY PRESERVATION FOR THE DISCRETE SYSTEM

Cosymmetric identity (1.5) is approximated by
\[
\sum_{j=1}^{m} \sum_{k=1}^{n} [\Phi_{1,j,k} g_{j,k} - \Phi_{2,j,k} f_{j,k}] = 0. \tag{3.1}
\]

Accordingly, the vector function $L_h = (g_{11}, \ldots, g_{1n}, \ldots, g_{nm}, f_{11}, \ldots, f_{1m}, \ldots, -f_{nm})$ obtained as an approximation of the cosymmetry $L = (\Psi, -\Theta)$ is the cosymmetry associated with discrete system (2.5).

Substituting (2.5) into (3.1) and using finite-difference formulas of integration by parts and Green’s theorem [8], we find that the contributions of the linear terms on the right-hand sides of (2.5) vanish. Consequently, (3.1) is equivalent to the following condition:
\[
\sum_{k=1}^{n} \sum_{j=1}^{m} (J_j g_j)_k = 0. \tag{3.2}
\]

Formula (3.2) is an approximation of identity (1.6), and the finite-difference counterpart of (1.7) is
\[
\sum_{k=1}^{n} \sum_{j=1}^{m} (J_j f_j)_k = 0. \tag{3.3}
\]

In what follows, we construct approximations of the expressions for $J_j$ so that both (3.2) and (3.3) are satisfied. Note that (3.2) and (3.3) are violated when the derivatives in $J_j$ are replaced by standard second-order accurate differences.
To approximate the nonlinear terms, we define the antisymmetric and symmetric differential operators

\[ D_a(f, g) = f'g - fg', \quad D_s(f, g) = f'g + fg' \]

and use them in the expression for \( J_x \):

\[
J_x = \frac{2\pi}{b} \sum_{i=1}^{m-j} \left( i + \frac{i}{2} \right) D_s(f_{i+p}, g_i) - \frac{i}{2} D_s(f_{i+p}, g_{i-j}) - \left( i + \frac{i}{2} \right) D_s(f_{i+p}, g_{i+j}) - \frac{i}{2} D_s(f_{i+p}, g_{i}) \]  

(3.4)

Let us construct special approximations of the operators \( D_a \) and \( D_s \) containing arbitrary parameters that can be manipulated so as to ensure that (3.2) and (3.3) hold.

**Lemma 1.** On a three-point stencil, there exists a one-parameter family of second-order accurate approximations of the antisymmetric operator \( D_a \) parameterized by \( \gamma \):

\[
d_{a, k}(f, g) = \left( \frac{1}{2h} + \gamma \right) \left( (f_{k+1} - f_{k-1})g_{k} - f_k(g_{k+1} - g_{k-1}) \right) + \gamma (f_{k-1}g_{k+1} - f_{k+1}g_{k-1}).
\]

(3.5)

**Proof.** The differential operator \( D_a \) is approximated by applying the method of undetermined coefficients:

\[ d_{a, k}(f, g) = \sum_{s, r = -1, 0, 1} p_{s, r} f_{k+s} g_{k+r}. \]

Expanding the right-hand side of this expression in the Taylor series about \( x_k \) up to the second order in \( h \), we obtain the following set of equations:

\[
\sum_{s, r = -1, 0, 1} p_{s, r} = 0, \quad \sum_{r = -1, 0, 1} (p_{1, r} - p_{-1, r}) = \frac{1}{h}, \quad \sum_{s = -1, 0, 1} (p_{s, 1} - p_{s, -1}) = -\frac{1}{h},
\]

\[
p_{-1, 1} + p_{1, 1} - p_{-1, 1} - p_{1, 1} = 0, \quad \sum_{r = -1, 0, 1} (p_{1, r} + p_{-1, r}) = 0, \quad \sum_{s = -1, 0, 1} (p_{1, s} + p_{s, -1}) = 0.
\]

The finite-difference operator \( d_a \) is antisymmetric if the coefficients \( p_{s, r} \) satisfy the following conditions:

\[ p_{-1, 1} = p_{0, 0} = p_{1, 1} = 0, \quad p_{-1, 0} = -p_{0, -1}, \quad p_{0, 1} = -p_{1, 0}, \quad p_{1, 1} = -p_{-1, 1}. \]

Thus, we have obtained a system of 12 equations for the unknown coefficients \( p_{s, r} (s, r = -1, 0, 1) \), whose solution is the one-parameter family

\[ p_{-1, 1} = p_{1, 1} = p_{0, 0} = 0, \quad p_{0, -1} = p_{1, 0} = -p_{-1, 0} = -p_{0, 1} = \frac{1}{2h} + \gamma, \quad p_{-1, 1} = -p_{1, -1} = \gamma. \]

Substituting the coefficients into the expression for \( d_{a, k} \), we obtain (3.5).

**Lemma 2.** On a three-point stencil, a second-order accurate approximation of the symmetric operator \( D_s \) is provided by the two-parameter family of finite-difference operators

\[
d_{s, k}(f, g) = a f_{k-1} g_{k-1} - \left( \frac{1}{2h} + \alpha + \beta \right) (f_{k-1} g_k + f_k g_{k-1}) + (6\beta - \alpha) f_k g_{k+1} \]

\[
+ \left( \frac{1}{2h} + \alpha - 3\beta \right) (f_k g_{k+1} + f_{k+1} g_k) + \beta (f_{k-1} g_{k+1} + 4f_k g_k + f_{k+1} g_{k-1}).
\]

(3.6)

The proof is analogous to that of Lemma 1.

**Proposition 1.** The finite-difference counterparts (3.2) and (3.3) of identities (1.6) and (1.7) are satisfied if the expressions in (3.4) are approximated by using finite-difference operators (3.5) and (3.6) with the following values of parameters: \( \gamma = 0, \alpha = -1/(3h), \) and \( \beta = 0. \)
**Proof.** Write discretized cosymmetric identity (3.2) for \( m = 2 \):

\[
\sum_{k=1}^{n} \left\{ d_s(f_1, g_1)g_2 + \frac{3}{2}[d_s(g_1, f_2)g_1]_k + \frac{1}{2}[d_s(g_1, f_2)g_1]_k \right. \\
\left. - \frac{3}{2}[d_s(g_2, f_1)g_1]_k - \frac{1}{2}[d_s(f_1, g_2)g_1]_k \right\} = 0.  
\]  

(3.7)

Substitute the expressions for \( d_s \) and \( d_t \) into (3.7), rearrange terms, and set to zero the coefficients of the third-order terms constructed from \( f_1, f_2, g_1, k \), and \( g_2, k \). As a result, (3.7) vanishes for arbitrary functions \( f_1 \) and \( f_2 \) if \( \alpha = -1/(3h) \) and \( \beta = 0 \). A similar analysis performed for finite-difference identity (3.3) shows that \( \gamma = 0 \).

Let us show that these values of \( \alpha, \beta, \gamma \) apply to the case of \( m > 2 \). When \( j < m \), relation (3.2) with nonzero \( \beta \) and \( \gamma \) contains the terms \(-5(\beta + \gamma)f_{2, k-1}g_{1, k+1}g_{1, k} \) and \(-3\gamma f_{2, k-1}g_{1, k+1}g_{1, k} \). Identity (3.2) is satisfied for arbitrary \( f_2, g_1 \), and \( g_3 \) if \( \beta = 0 \) and \( \gamma = 0 \). Note also that the expression for \( d_s \) is not invariant with respect to the change \( h \to -h \) if \( \beta \neq 0 \). Thus, it remains to check that the value of \( \alpha \) remains equal to \(-1/(3h)\) for any number of harmonics. Rewrite (3.2) as follows:

\[
\frac{\pi(1 + 3\alpha h)}{bh} \sum_{k=1}^{n} \sum_{j=1}^{m} f_{j,k}Q_{j,k} = 0, 
\]

where

\[
Q_{j,k} = \sum_{i=1}^{j-1} jg_{i,k}(g_{j-1,k+1} - g_{i,k}g_{j-1,k-1}) + \sum_{v=-1,1} \sum_{i_1+i_2 = j} j\nu g_{i_1,k+\nu}g_{i_2,k+\nu} - R_{j,k} \\
+ j\sum_{i=1}^{m-j} [g_{i+j,k+1}(g_{i,k+1} - g_{i,k}) - g_{i+j,k-1}(g_{i,k-1} - g_{i,k}) + g_{i+j,k}(g_{i,k-1} - g_{i,k+1})]. 
\]

\( R_{j,k} \) is zero for odd \( j \) and \( R_{2l,k} = l(g_{2l,k+1}^2 - g_{2l,k-1}^2) \) for \( j = 2l \).

Since \( f_{j,k}, g_{i,k+s} \) and \( g_{f+1,k+s} (s = -1, 0, 1) \) are arbitrary grid functions, it follows that \( \alpha = -1/(3h) \).

The operators \( d_s \) and \( d_t \) with \( \gamma = 0, \alpha = -1/(3h), \) and \( \beta = 0 \) ensure that (3.2) and (3.3) hold for any number \( m \) of harmonics:

\[
d_{s,k}(f, g) = \frac{f_{k+1} - f_{k-1}}{2h} g_k - f_k \frac{g_{k+1} - g_{k-1}}{2h}, \\
d_{t,k}(f, g) = \frac{2f_{k+1}g_{k+1} + 2f_{k+1}g_{k+1} - 2f_{k+1}g_{k+1}}{6h}. 
\]

4. METHOD FOR COMPUTING A FAMILY OF STEADY-STATE SOLUTIONS

Rewrite system (2.5) in vector form:

\[
\frac{d}{dt} F = AF + \lambda BG + L(F, G), \quad 0 = AG - BF, 
\]

(4.1)

where \( F = (f_{11}, \ldots, f_{n1}, \ldots, f_{1m}, \ldots, f_{nm}), G = (g_{11}, \ldots, g_{n1}, \ldots, g_{1m}, \ldots, g_{nm}), \)

\[
A = \begin{pmatrix} A_1 & 0 \\ \vdots & \ddots \\ 0 & A_m \end{pmatrix}, \quad A_j = \begin{pmatrix} -2c_jh^2 & 1 & 0 & 0 \\ 1 & -2c_jh^2 & 1 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 1 & -2c_jh^2 \end{pmatrix}. 
\]
The matrix $A$ is constructed from $m$ tridiagonal submatrices $A_j$ of dimension $n$, the nonzero entries of the skew-symmetric matrix $B = \{ b_{sr} \}_{s,r=1}^{nm}$ are $b_{s,s+1} = -b_{1+s,1} = h/2 \ (s = 1, nm - 1)$, and $L(F, G)$ denotes the nonlinear terms in Eq. (2.5).

To integrate system (4.1), we wrote a MATLAB code which can be used to compute convection dynamics starting from arbitrary initial conditions, to calculate the family of steady regimes, and to analyze the temperature and streamfunction fields.

The computation of families of steady-state solutions is based on the cosymmetric version of the implicit function theorem [4]. As the parameter $\lambda$ passes through the critical point $\lambda_{11}$, which corresponds to the instability threshold for the zero equilibrium, a family of steady regimes (equilibria) of system (4.1) branches off. It was shown in [3] that all equilibria in the family are stable when $\lambda$ is slightly greater than $\lambda_{11}$. Thus, starting from a neighborhood of the unstable zero equilibrium and integrating system (4.1), we can obtain a stable equilibrium point $F_0$ belonging to the family. The integration is performed by the fourth-order Runge–Kutta method, and the family is calculated by using the following algorithm.

**Step 1.** Determine the point $F_0$ by Newton’s method, calculating the Jacobian (linearization) matrix numerically. In the neighborhood of the family, use modified Newton’s method to deal with degeneracy.

**Step 2.** Find the kernel of the linearization matrix by SVD decomposition and predict the value of the next point in the family by the Adams extrapolation method.

This algorithm was tested by computing the families for problem (1.1), (1.4) in [9], where discretization with respect to both coordinates was based on a finite-difference method. Originally, the steady-state solutions were calculated in [17, 18], where Galerkin’s method was applied and the Jacobian matrix was determined analytically.

### 5. Numerical Results

We used the algorithm described above to calculate families containing stable and unstable equilibria, to determine the bifurcation points at which unstable arcs appear in a family, and to examine some scenarios of breakup and merger of solution families.

We present here the numerical results obtained for the family of equilibria corresponding to a narrow rectangle with aspect ratio $b/a = 1.5$. The critical point for the first transition (from the quiescent state) was found by linearizing system (4.1) about the zero equilibrium $F = G = 0 \ (f_{jk} = g_{jk} = 0, j = 1, m, k = 1, n)$ and solving the corresponding spectral problem.

The table shows the first critical values of $\lambda$ calculated for various numbers $n$ of grid points on the $x$-axis with the use of various numbers $m$ of $y$-harmonics. Here, $a = 2$ and $b = 3$.

With increasing $m$ and $n$, the calculated values of $\lambda$ monotonically approach the corresponding exact values (shown in the bottom row). To guarantee the required accuracy, the number $n$ of grid points should be greater than the number $m$ of harmonics. It is clear that the $16 \times 6$ discretization ensures that the errors of the first three critical points are within 5%. Further computations showed that discretizations with a moderate number of unknowns (with a product of $m$ with $n$ about 100) and $n > m$ are sufficient even in determining the critical point for the second transition, when unstable equilibria appear in the family.

Figures 1–6 show the numerical results obtained for the families corresponding to various Rayleigh numbers. A family is depicted by using the following integral characteristics introduced in [17]: the cumulative

<table>
<thead>
<tr>
<th>$n \times m$</th>
<th>$\lambda_{11}$</th>
<th>$\lambda_{12}$</th>
<th>$\lambda_{13}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>12 $\times$ 6</td>
<td>14.950</td>
<td>29.362</td>
<td>54.589</td>
</tr>
<tr>
<td>16 $\times$ 6</td>
<td>14.655</td>
<td>28.533</td>
<td>52.352</td>
</tr>
<tr>
<td>24 $\times$ 12</td>
<td>14.438</td>
<td>27.925</td>
<td>50.716</td>
</tr>
<tr>
<td>30 $\times$ 20</td>
<td>14.374</td>
<td>27.745</td>
<td>50.232</td>
</tr>
<tr>
<td>40 $\times$ 20</td>
<td>14.323</td>
<td>27.603</td>
<td>49.852</td>
</tr>
<tr>
<td>Exact value</td>
<td>14.256</td>
<td>27.415</td>
<td>49.348</td>
</tr>
</tbody>
</table>
heat flux $Nu_v$ across the bottom boundary and the cumulative horizontal heat flux $Nu_h$ across the vertical median line of the rectangle:

$$Nu_h = \int_0^b \frac{\partial \Theta}{\partial x} \bigg|_{x=a/2} dy, \quad Nu_v = \int_0^a \frac{\partial \Theta}{\partial y} \bigg|_{y=0} dx.$$

Figure 1 shows the numerical results obtained for the family corresponding to $\lambda = 20$ by the spectral–difference method with $n = 12$ and $m = 6$ (curve 1) and with $n = 16$ and $m = 8$ (curve 2) and by the finite-difference method developed in [9] on a $12 \times 8$ grid (dotted curve). Note that good agreement between the results is achieved on a relatively coarse grid. In [9], an approximation of the Jacobian proposed in [10, 11] was used to preserve cosymmetry in a grid discretization of problem (1.1), (1.2).

Figure 2 shows the families of equilibria computed for various values of the Rayleigh number $\lambda$ with $m = 6$ and $n = 16$. With increasing $\lambda$, the family distorts, and unstable equilibria emerge in it at $\lambda = 38.5$ through monotonic instability. Thus, the second transition through monotonic instability of the family can occur in the case of a narrow container ($b > a$), as well as in the case of a wide container ($b < a$) (see [17]). Computations performed for rectangles with $b/a \leq 2.2$ showed that the first loss of the family’s stability is due to monotonic instability.

Figure 3 shows the equilibrium spectra corresponding to the families computed for $\lambda = 35$ and $\lambda = 40$ with $n = 16$ and $m = 6$. Here, the ordinate axis is the real part of an eigenvalue, and the abscissa axis is the number of an equilibrium parameterized by the angle $\varphi \in [0, 2\pi]$ associated with the mapping of the family to a circle. When $\lambda = 35$, the entire family is stable. When $\lambda = 40$, the family contains two arcs consisting of unstable equilibria. The spectrum of equilibria varies with the location of the state point in the family, which supports the observation that the emergence of a family is not related to any symmetry in the problem under analysis (see [3]).

As the state point moves along the family, the steady convection regime transforms continuously. Figure 4a depicts the families computed for $\lambda = 20$ and $\lambda = 35$ in the plane of $Nu_h$ and $Nu_v$. The symbols denoted by letters correspond to the steady regimes for which streamlines and eigenvalue distributions are shown in the upper and lower panels of Figs. 5 and 6. Figure 4b shows the same families projected onto the two-dimensional unstable manifold associated with the zero equilibrium for $\lambda = 20$ (in the plane of coordinates
$U_1$ and $U_2$ parameterizing the manifold). In this projection, the family grows with increasing $\lambda$, preserving its symmetry with respect to $U_1$ and $U_2$.

The regimes denoted by $a$ and $d$ in Fig. 5 and by $A$ and $D$ in Fig. 6 are characterized by the lowest and highest values of heat flux across the central vertical cross section of the container. In these regimes, the streamlines and temperature distributions are symmetric about the cross section. The state points lying between $a$ and $d$ (A and D) represent asymmetric steady regimes, and the equilibrium $c$ (C) is characterized by the highest value of heat flux across the bottom side of the container. In this regime, the central convection roll shrinks with increasing $\lambda$, while two corner rolls grow. The changes observed in the eigenvalue patterns (bottom row) as $\lambda$ is increased are associated with increase in the imaginary component. Note that the highest concentration of eigenvalues at the real axis is characteristic of symmetric regimes.

When Eqs. (2.5) do not retain the symmetry of the original system, a finite number of isolated equilibria branch off from the zero equilibrium (quiescent state) in computations at the first transition point, instead of a family of steady regimes. For example, when the nonlinear terms in (3.4) are approximated by (3.5) and (3.6) with $\gamma = 0$, $\beta = 0$, $g = 0$, and $b = 0$,
and $\alpha = -\delta/(3h)$ (where $\delta \neq 1$), the following scenario is typically observed: as $\lambda$ passes through $\lambda_{11}$, the zero equilibrium becomes unstable, and four equilibria emerge. Two of them are symmetric about the line $x = a/2$ and are represented by points with zero values of $Nu_h$ in the plane of $Nu_h$ and $Nu_v$, whereas the other two equilibria are asymmetric. The symmetric and asymmetric regimes are stable if $\delta > 1$ and $\delta < 1$, respectively. When $\delta = 1$, a family of steady regimes is obtained. In particular, computations performed with $n = 12$ and $m = 6$ for $\delta = -2$ showed that the pair of asymmetric equilibria remains stable until $\lambda = 66$ is reached, whereas unstable equilibria emerge in the family at $\lambda = 38.5$. Figure 7 depicts the family corresponding to
\( \lambda = 20, \delta = 1 \) in the plane of \( Nu_h \) and \( Nu_v \). Here, asterisks and circles (\( \delta = -2 \)) represent stable and unstable equilibria, respectively. It is clear that the deviations of the isolated equilibria relative to the family are insignificant. When equilibrium points of the family are taken as initial points, and the computation is based on formulas that do not preserve the cosymmetry, the relaxation toward a steady state is extremely slow. The corresponding trajectory passes near the family before approaching an isolated equilibrium; i.e., the finite-dimensional system keeps some “memory” of the family.

Thus, when approximations that do not retain the cosymmetry are used, the phase portrait changes: computations result in several isolated equilibria instead of a continuous family of steady-state solutions. Moreover, the inference about stability of various (symmetric or asymmetric) regimes depends on the value of the parameter contained in the finite-difference expression approximating the convective terms.

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